The Advantage Testing Foundation

Math Prize for Girls at MIT

2017 Solutions

Problem 1 A bag contains 4 red marbles, 5 yellow marbles, and 6 blue marbles. Three marbles are to be picked out randomly (without replacement). What is the probability that exactly two of them have the same color? Express your answer as a fraction in simplest form.

Answer: $\dfrac{43}{65}$

Solution: Since all outcomes are equally likely, the probability is the ratio of the number of favorable outcomes to the total number of ways 3 marbles can be drawn. Here, a favorable outcome is one in which there are exactly 2 marbles of the same color.

Before we start counting, we should decide whether we care about the order in which marbles are drawn. For this problem, it doesn't matter which way you decide; you just have to be consistent.

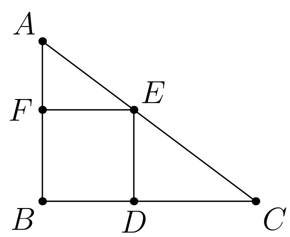
If you decide not to worry about order, then there are $\binom{15}{3}$ ways that 3 marbles can be drawn. The favorable outcomes can be split into 3 cases depending on which color is repeated. There are $11\binom{4}{2}$ ways to have 2 red marbles, $10\binom{5}{2}$ ways to have 2 yellow marbles, and $9\binom{6}{2}$ ways to have 2 blue marbles. Therefore, the answer is

$$\frac{11\binom{4}{2} + 10\binom{5}{2} + 9\binom{6}{2}}{\binom{15}{3}} = \frac{11 \cdot 6 + 10 \cdot 10 + 9 \cdot 15}{\frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1}} = \boxed{\frac{43}{65}}.$$

If you decide to keep track of order, then there are $15 \cdot 14 \cdot 13$ ways that 3 marbles can be drawn. The favorable outcomes can be split into 3 cases depending on which color is repeated. There are $3 \cdot 11 \cdot (4 \cdot 3)$ ways to have 2 red marbles, $3 \cdot 10 \cdot (5 \cdot 4)$ ways to have 2 yellow marbles, and $3 \cdot 9 \cdot (6 \cdot 5)$ ways to have 2 blue marbles. (The first factor of 3 in each of these products accounts for the 3 different ways the 3 marbles with exactly 2 of the same color can be ordered.) Therefore, the answer is

$$\frac{3 \cdot 11 \cdot (4 \cdot 3) + 3 \cdot 10 \cdot (5 \cdot 4) + 3 \cdot 9 \cdot (6 \cdot 5)}{15 \cdot 14 \cdot 13} = \boxed{\frac{43}{65}}.$$

Problem 2 In the figure below, $BDEF$ is a square inscribed in $\triangle ABC$.



If $\frac{AB}{BC} = \frac{4}{5}$, what is the area of $BDEF$ divided by the area of $\triangle ABC$? Express your answer as a fraction in simplest form.

Answer: $\dfrac{40}{81}$

Solution: Observe that $\triangle AFE$ and $\triangle EDC$ are similar to $\triangle ABC$ (because they share the same angle measurements). Note that E is the foot of the angle bisector from vertex B. By the angle bisector theorem, $\frac{AE}{EC} = \frac{AB}{BC} = \frac{4}{5}$. Therefore $\frac{AE}{AC} = \frac{4}{4+5} = \frac{4}{9}$ and the ratio of the area of $\triangle AFE$ to the area of $\triangle ABC$ is $(\frac{4}{9})^2$. Likewise, the ratio of the area of $\triangle EDC$ to the area of $\triangle ABC$ is $(\frac{5}{9})^2$. Thus, the desired ratio is $1 - (\frac{4}{9})^2 - (\frac{5}{9})^2 = \boxed{40/81}$.

An alternative way to find $\frac{AE}{EC}$ that avoids the angle bisector theorem is to use similarity and the fact that $FE = ED$ to see that $\frac{4}{5} = \frac{AB}{BC} = \frac{AF}{FE} = \frac{AF}{ED} = \frac{AE}{EC}$.

Remark. In any triangle ABC, the side length of the largest rhombus that shares the corner of the triangle around vertex B is half the harmonic mean of AB and BC. Knowing this, the problem can be solved by assuming $AB = 4$ and $BC = 5$ and computing $BD = \frac{1}{\frac{1}{4}+\frac{1}{5}} = \frac{20}{9}$, so the desired ratio is $(\frac{20}{9})^2/(\frac{1}{2}4 \cdot 5) = \boxed{40/81}$.

Problem 3 If A and B are numbers such that the polynomial $x^{2017} + Ax + B$ is divisible by $(x+1)^2$, what is the value of B?

Answer: -2016

Solution: We are given that $x^{2017} + Ax + B = (x+1)^2 p(x)$ for some polynomial $p(x)$. Differentiating, we get $2017x^{2016} + A = 2(x+1)p(x) + (x+1)^2 p'(x)$. Substituting -1 into these equations gives us a system of two equations in A and B: $-1 - A + B = 0$ and $2017 + A = 0$. Solving for B, we find $B = \boxed{-2016}$.

If you don't know how to differentiate polynomials, we can proceed as follows. First, substitute -1 into $x^{2017} + Ax + B = (x+1)^2 p(x)$ to see that $-1 - A + B = 0$. Thus,

$$(x+1)^2 p(x) = x^{2017} + (B-1)x + B = x^{2017} - x + B(x+1).$$

Divide both sides by $x + 1$ to get

$$(x+1)p(x) = B + \sum_{k=1}^{2016} (-1)^k x^k.$$

When we substitute -1 for x in this last equation, we find $B = \boxed{-2016}$.

Remark. A polynomial p has multiple roots if and only if it has a nontrivial common factor with its (formal) derivative.

Problem 4 If MATH + WITH = GIRLS, compute the smallest possible value of GIRLS. Here MATH and WITH are 4-digit numbers and GIRLS is a 5-digit number (all with nonzero leading digits). Different letters represent different digits.

Answer: 10,358

Solution: First, we make a few general remarks. Since we don't care what MATH and WITH are and addition is commutative, we only have to concentrate on minimizing GIRLS subject to the constraint that GIRLS is a sum of two 4-digit numbers that are equal modulo 100 (because of TH) and otherwise have different digits. Also, to minimize GIRLS, we should aim to minimize the digits from most significant to least significant. We're told that leading digits are nonzero, and since the sum of two d-digit numbers is less than 2×10^d, we must have G = 1 and any carries will be 0 or 1.

Note that S must be even since it is the units digit of $2H$.

There is a lot of freedom in digit choices, so we try to find a solution where I = 0.

The smallest unused digit would then be 2, so we look for a solution where R = 2. This, however, is impossible since if R = 2 then A must be 1 or 2, both of which are already taken.

So we try to find a solution where R = 3. This forces A to be 2. With A = 2 there can be no carry to the thousands place, hence $\{\mathrm{M}, \mathrm{W}\} = \{4, 6\}$. This leaves 8 for S, which forces H = 9. That leaves the digits 5 and 7 for T and L. Only T = 7 and L = 5 works. Hence the solution is $\boxed{10{,}358}$.

Problem 5 The New York Public Library requires patrons to choose a 4-digit Personal Identification Number (PIN) to access its online system. (Leading zeros are allowed.) The PIN is not allowed to contain either of the following two forbidden patterns:

- A digit that is repeated 3 or more times in a row. For example, 0001 and 5555 are not PINs, but 0010 is a PIN.
- A pair of digits that is duplicated. For example, 1212 and 6363 are not PINs, but 1221 and 6633 are PINs.

How many distinct possible PINs are there?

Answer: 9720

Solution: Let S be the set of 4-digit sequences that include a digit repeated 3 or more times. Let T be the set of 4-digit sequences where a pair of digits is duplicated, as explained in the problem statement. We count the number of elements in $S \cup T$.

Elements in S are of the form XXXY, XYYY, or XXXX, where X and Y are distinct digits. There are a total of 10×9, 10×9, and 10 of each of these types, respectively, for a total of 190 elements in S.

Elements in T are of the form XYXY, for distinct X and Y, or XXXX. There are 10×9 of the first type and 10 of the second type for a total of 100 elements in T.

Elements in $S \cap T$ are of the form XXXX, so $S \cap T$ has 10 elements.

Therefore, $S \cup T$ contains $190 + 100 - 10 = 280$ elements, by the principle of inclusion/exclusion.

Since there are 10^4 4-digit sequences, the number of PINs is equal to $10^4 - 280 = \boxed{9720}$.

Problem 6 Let b and c be integers chosen randomly (uniformly and independently) from the set

$$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}.$$

(Note that b and c can be equal.) What is the probability that the two roots of the quadratic $x^2 + bx + c$ are consecutive integers? Express your answer as a fraction in simplest form.

Answer: $\dfrac{6}{169}$

Solution: Because b and c are chosen uniformly and independently at random, the answer is the number of pairs (b, c) such that the roots of $x^2 + bx + c$ are consecutive integers divided by the total number of pairs (b, c). There are a total of $13^2 = 169$ pairs. The condition that a monic quadratic have roots that are consecutive integers is a strong condition, so rather than checking if pairs (b, c) yield monic quadratics with consecutive integer roots, we determine the form of such a quadratic and see which have coefficients belonging to the given set. Such a quadratic has the form $(x - r)(x - (r + 1)) = x^2 - (2r + 1)x + r(r + 1)$, where r is an integer. For the coefficient of x to be in the given set, we need $|2r + 1| \le 6$, that is $r = -3, -2, -1, 0, 1, 2$. All 6 of these choices satisfy $|r(r + 1)| \le 6$, hence the answer is $\boxed{6/169}$.

Problem 7 Let a_1, a_2, $\ldots$ be an infinite sequence of integers such that $0 \le a_k \le k$ for every positive integer k and such that

$$2017 = \sum_{k=1}^{\infty} a_k \cdot k!\,.$$

What is the value of the infinite series $\sum_{k=1}^{\infty} a_k$?

Answer: 11

Solution: Representing numbers in the manner described is highly analogous to the standard way of representing numbers using decimal notation where the a_k are the analogue of decimal digits.

Note that $(n + 1)! - 1 = (n + 1) \cdot n! - 1 = n \cdot n! + n! - 1$. Hence, using recursion we see that $(n + 1)! - 1 = \sum_{k=1}^{n} k \cdot k!$. Therefore, to represent a whole number N in the form $\sum_{k=1}^{\infty} a_k \cdot k!$ where $0 \le a_k \le k$, we have to choose the a_k greedily, starting with the largest k for which $k! \leq N$, for if we don't, we will not be able to overcome the shortfall produced even if we let $a_i = i$ for all lower indices.

Since $6! = 720 < 2017$ and $7! = 5040 > 2017$, we have $a_k = 0$ for all $k > 6$.

The largest multiple of $6!$ that does not exceed 2017 is $2 \cdot 6!$, so $a_6 = 2$, and $2017 - 2 \cdot 6! = 577$.

The largest multiple of $5! = 120$ that does not exceed 577 is $4 \cdot 5!$, so $a_5 = 4$, and $577 - 4 \cdot 5! = 97$.

The largest multiple of $4! = 24$ that does not exceed 97 is $4 \cdot 4!$, do $a_4 = 4$, and $97 - 4 \cdot 4! = 1$.

Thus, $a_3 = a_2 = 0$ and $a_1 = 1$. Hence, $\sum_{k=1}^{\infty} a_k = 1 + 4 + 4 + 2 = \boxed{11}$.

An alternative solution is to start with a_1. Looking mod $2!$, we get $a_1 = 1$. Mod $3!$, we get $a_2 = 0$. Mod $4!$, we get $a_3 = 0$. Mod $5!$, we get $a_4 = 4$. Mod $6!$, we get $a_5 = 4$. Mod $7!$, we get $a_6 = 2$. So the sum of the a_k is $\boxed{11}$.

Problem 8 Let c be a complex number. Suppose there exist distinct complex numbers r, s, and t such that for every complex number z, we have

$$(z - r)(z - s)(z - t) = (z - cr)(z - cs)(z - ct).$$

Compute the number of distinct possible values of c.

Answer: 4

Solution: Let $p(z) = (z - r)(z - s)(z - t) \equiv z^3 + Az^2 + Bz + C$. The condition is equivalent to $p(z) = c^3 p(z/c)$.

We compute $c^3 p(z/c) = z^3 + cAz^2 + c^2 Bz + c^3 C$. By comparing coefficients, we must have $cA = A$, $c^2 B = B$, and $c^3 C = C$.

If $A \neq 0$, then $c = 1$, and, in fact, $c = 1$ works for any r, s, and t.

If $A = 0$ but $B \neq 0$, then $c^2 = 1$. We already know c can be 1. If $c = -1$, then we also need $C = 0$, and $p(z) = z^3 + Bz$. The roots of this polynomial are 0 and $\pm\sqrt{-B}$ which are distinct (recall $B \neq 0$). Hence -1 is a possible value of c.

If $A = 0$ and $B = 0$, then $p(z) = z^3 + C$, which has roots v, wv, and $w^2 v$, where v is any cube root of $-C$ and $w = e^{2\pi i/3}$. These roots are distinct provided $C \neq 0$. For such a polynomial, c equal to 1, w, or w^2 all work.

We cannot have $A = B = C = 0$, for then p does not have distinct roots. We conclude that the possible values of c are 1, -1, w, and w^2. So the number of possible values of c is $\boxed{4}$.

Problem 9 Say that a positive integer n is *smooth* if $\frac{1}{n}$ has a terminating decimal expansion. (Note that 1 is smooth.) Compute the value of the infinite series

$$\sum_n \frac{1}{n^3},$$

where n ranges over all smooth positive integers. Express your answer as a fraction in simplest form.

Answer: $\dfrac{250}{217}$

Solution: The fraction $\frac{1}{n}$ has a terminating decimal expansion if and only if $n = 2^k 5^j$ for nonnegative integers k and j. (To see why, suppose that $\frac{1}{n}$ has a terminating decimal expansion. Then it is equal to a fraction of the form $\frac{a}{10^m}$, where a is an integer. Hence $a = \frac{10^m}{n}$, which shows that n divides evenly into 10^m and must therefore be of the form $2^k 5^j$ for nonnegative integers k and j. Conversely, if $n = 2^k 5^j$ for nonnegative integers k and j, then $\frac{1}{n} = \frac{1}{2^k 5^j} = \frac{2^{M-k} 5^{M-j}}{10^M}$, where M is the larger of k and j.)

We compute

$$\begin{aligned}
\sum_{n \text{ smooth}} \frac{1}{n^3} &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(2^k 5^j)^3} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^{3k} 5^{3j}} \\
&= \Big(\sum_{k=0}^{\infty} \frac{1}{8^k}\Big)\Big(\sum_{j=0}^{\infty} \frac{1}{125^j}\Big) \\
&= \frac{1}{1 - 1/8} \cdot \frac{1}{1 - 1/125} \\
&= \boxed{\frac{250}{217}}.
\end{aligned}$$

Problem 10 Let C be a cube. Let P, Q, and R be random vertices of C, chosen uniformly and independently from the set of vertices of C. (Note that P, Q, and R might be equal.) Compute the probability that some face of C contains P, Q, and R. Express your answer as a fraction in simplest form.

Answer: $\dfrac{37}{64}$

Solution: Since a cube has 8 vertices, each vertex has a $1/8$ probability of being chosen as P.

There is a $1/8$ chance that $Q = P$, a $3/8$ chance that Q is connected to P by an edge, a $3/8$ chance that Q is diagonally across a face from P, and a $1/8$ chance that Q is diagonally across the cube from P. We treat these 4 possibilities as separate cases.

If $Q = P$, then the only vertex R that is not contained in a common face with Q and P is the vertex diagonally across the cube from $Q = P$. So there is a $7/8$ chance that P, Q, and R are contained in a common face.

If Q is connected to P by an edge, then the only vertices R that are not contained in a common face with Q and P is one of the two vertices that are the endpoints of the edge that is opposite the edge $\overline{PQ}$ in the cube. So there is a $6/8$ chance that P, Q, and R are contained in a common face.

If Q is diagonally across a face from P, then there is a unique face that contains both P and Q, so there is a $4/8$ chance that P, Q, and R are contained in a common face.

If Q is diagonally across the cube from P, there is no face that contains P and Q and hence, there is no chance that P, Q, and R are contained in a common face.

Thus, the desired probability is $8 \times \frac{1}{8}(\frac{1}{8} \cdot \frac{7}{8} + \frac{3}{8} \cdot \frac{6}{8} + \frac{3}{8} \cdot \frac{4}{8} + \frac{1}{8} \cdot \frac{0}{8}) = \boxed{37/64}$.

Problem 11 Let $S(N)$ be the number of 1's in the binary representation of an integer N, and let $D(N) = S(N+1) - S(N)$. Compute the sum of $D(N)$ over all N such that $1 \le N \le 2017$ and $D(N) < 0$.

Answer: -1002

Solution: Look at the binary representation of a whole number n and consider the longest (possibly empty) block B of consecutive 1's that contains the units digit. The binary representation of $n + 1$ will look just like that of n except that the digits in B will be replaced with 0's and the 0 just to the left of B will be replaced with a 1. Therefore, the number of 1's in the binary representation of $n+1$ will be less than the number of 1's in the binary representation of n whenever B consists of 2 or more binary digits, that is, when $n \equiv 3 \pmod 4$. Therefore, the desired sum is

$$\sum_{\substack{n\equiv 3 \ (\mathrm{mod}\ 4)\\ 1\le n\le 2017}} (S(n+1)-S(n)) = \sum_{\substack{n\equiv 0 \ (\mathrm{mod}\ 4)\\ 1\le n\le 2017}} S(n) - \sum_{\substack{n\equiv 3 \ (\mathrm{mod}\ 4)\\ 1\le n\le 2017}} S(n).$$

Now observe that $S(4k) - S(4k+3) = -2$. Thus,

$$\begin{aligned}\sum_{\substack{n\equiv 0 \ (\mathrm{mod}\ 4)\\ 1\le n\le 2017}} S(n) - \sum_{\substack{n\equiv 3 \ (\mathrm{mod}\ 4)\\ 1\le n\le 2017}} S(n) &= S(2016) - S(3) + \sum_{i=1}^{503}(S(4i) - S(4i+3)) \\ &= S(2016) - S(3) - 2\cdot 503.\end{aligned}$$

Since $2016 = 11111100000_2$ and $3 = 11_2$, the answer is $6 - 1006 - 2 = \boxed{-1002}$.

Problem 12 Let S be the set of all real values of x with $0 < x < \pi/2$ such that $\sin x$, $\cos x$, and $\tan x$ form the side lengths (in some order) of a right triangle. Compute the sum of $\tan^2 x$ over all x in S. Express your answer in simplified radical form.

Answer: $\sqrt{2}$

Solution: Because $\sin x < \tan x$ for $0 < x < \pi/2$, the hypotenuse can't have length $\sin x$. So the length of the hypotenuse is either $\tan x$ or $\cos x$.

If $\tan x$ is the length of the hypotenuse, then $\sin^2 x + \cos^2 x = \tan^2 x$, by the Pythagorean theorem. Since $\sin^2 x + \cos^2 x = 1$, we must have $\tan x = \pm 1$. The only solution to this with $0 < x < \pi/2$ is $x = \pi/4$.

If $\cos x$ is the length of the hypotenuse, then $\sin^2 x + \tan^2 x = \cos^2 x$. Expressing everything in terms of $\cos x$ and simplifying, this equation becomes $1 - \cos^2 x + 1/\cos^2 x - 1 = \cos^2 x$ or $1/2 = \cos^4 x$. For $0 < x < \pi/2$ this equation has one solution (since $\cos^4 x$ decreases monotonically from 1 to 0 on that interval), and for this solution, $\tan^2 x = 1/\cos^2 x - 1 = \sqrt{2} - 1$.

The answer is therefore $\tan^2(\pi/4) + \sqrt{2} - 1 = \boxed{\sqrt{2}}$.

Problem 13 A polynomial whose roots are all equal to each other is called a *unicorn*. Compute the number of distinct ordered triples (M, P, G), where M, P, G are complex numbers, such that the polynomials $z^3 + Mz^2 + Pz + G$ and $z^3 + Gz^2 + Pz + M$ are both unicorns.

Answer: 5

Solution: A monic cubic unicorn has the form $(z-r)^3 = z^3-3rz^2+3r^2z-r^3$. Let r be the unique root of z^3+Mz^2+Pz+G and let R be the unique root of z^3+Gz^2+Pz+M. By comparing coefficients, we see that $M = -3r = -R^3$, $P = 3r^2 = 3R^2$, and $G = -r^3 = -3R$. From the expressions for P, we see that $r = \pm R$. If $r = R$, then from the expression for M or G, we find $3r = r^3$, so $r = 0, \pm\sqrt{3}$, which yield the triples $(3\sqrt{3}, 9, 3\sqrt{3})$, $(-3\sqrt{3}, 9, -3\sqrt{3})$, and $(0, 0, 0)$. If $r = -R$, then from the expression for M or G, we find $3r = -r^3$, so $r = 0, \pm\sqrt{3}i$, which yield the triples $(3\sqrt{3}i, -9, -3\sqrt{3}i)$, $(-3\sqrt{3}i, -9, 3\sqrt{3}i)$, and $(0, 0, 0)$.

In total, the number of triples is $\boxed{5}$.

Problem 14 A *permutation* of a finite set S is a one-to-one function from S to S. Given a permutation f of the set $\{1, 2, \ldots, 100\}$, define the *displacement* of f to be the sum

$$\sum_{i=1}^{100} |f(i) - i| .$$

How many permutations of $\{1, 2, \ldots, 100\}$ have displacement 4?

Answer: 5047

Solution: Let f be a permutation with displacement 4.

We consider cases that are defined by the number of elements in the set $D \equiv \{i | f(i) \neq i\}$. Note that $D \neq \emptyset$, since otherwise $f(i) = i$ for all i and the displacement would be 0.

If $f(i) \neq i$, then $|f(i) - i| \geq 1$. Therefore, D cannot have more than 4 elements. Also, note that f maps D to itself. In particular, D cannot consist of a single element. So we have 3 cases to consider: either D has 2, 3, or 4 elements.

Suppose D has 2 elements $a < b$. Then $f(a) = b$ and $f(b) = a$. Hence $|f(a) - a| + |f(b) - b| = 2(b - a) = 4$. Thus, $b = a + 2$ and a can be any number from 1 to 98. We get 98 such permutations.

Suppose D has 3 elements $a < b < c$. The only way for 3 positive integers to add up to 4 is to have two 1's and one 2. Since none of a, b, or c is fixed by f, they must be cyclically permuted, so either $f(a) = b$, $f(b) = c$, and $f(c) = a$ or $f(a) = c$, $f(b) = a$, and $f(c) = b$. In either case, the biggest individual displacement is $c - a$ which must therefore equal 2, and, hence, a, b, and c are consecutive. There are 98 groups of 3 consecutive integers from among $\{1, 2, \ldots, 100\}$, and we get 2 permutations for each such group for a total of 196 such permutations.

Finally, suppose D has 4 elements $a < b < c < d$. The only way for 4 positive integers to add up to 4 is for all 4 to be equal to 1. Therefore, each of a, b, c, and d must be sent to an adjacent number. Thus, $f(a) = b = a+1$ and $f(d) = c = d - 1$. Now, $f(b)$ must go to either a or c, but since $f(d) = c$, we must have $f(b) = a$. Similarly, $f(c) = d$. We conclude that the number of such permutations is equal to the number of nonoverlapping pairs of consecutive integers from among $\{1, 2, \ldots, 100\}$. Because $2 \le b < c \le 99$, there are $\binom{98}{2}$ choices for the pair $\{b, c\}$. So the number of permutations in this case is $\binom{98}{2} = 4753$.

In total, the number of such permutations is $98 + 196 + \binom{98}{2} = \boxed{5047}$.

Problem 15 A restricted rook (RR) is a fictional chess piece that can move horizontally or vertically (like a rook), except that each move is restricted to a neighboring square (cell). If RR can only (with at most one exception) move up and to the right, how many possible distinct paths are there to move RR from the bottom left square to the top right square of a standard 8-by-8 chess board? Note that RR may visit some squares more than once. A path is the sequence of squares visited by RR on its way.

Answer: 163,592

Solution: We can describe RR's moves by a word made from the 4 letters U, R, D, L, where each letter corresponds to a move up, right, down, or left, respectively. Every such word corresponds to a different path, although some paths may not be valid because they travel off the board or do not end in the appropriate square.

If RR only moves up and right, then the word will consist of 7 U's and 7 R's in any order. This yields $\binom{14}{7}$ paths.

Suppose RR moves left once along its journey. Then RR's moves will be described by a word with 7 U's, 8 R's, and 1 L. Note that the L must not occur before or after all the R's since RR may not go off the chess board. Once the position of L among the R's is fixed, the U's can be thrown in at will. There are 7 strings of 8 R's and 1 L with the L not at the ends, and we can place the U's into this word in $\binom{7+9}{9}$ ways (think of the 7 U's as "balls" and the 10 spaces before, after, and between the R's and L as "urns"), for a total of $7 \cdot \binom{16}{9}$ paths.

The case where RR moves down once is symmetric to the case where RR moves left once via reflection over the main diagonal connecting start and finish.

In total, the number of paths is $2 \cdot 7 \cdot \binom{16}{9} + \binom{14}{7} = \boxed{163{,}592}$.

Problem 16 Samantha is about to celebrate her sweet 16th birthday. To celebrate, she chooses a five-digit positive integer of the form SWEET, in which the two E's represent the same digit but otherwise the digits are distinct. (The leading digit S can't be 0.) How many such integers are divisible by 16?

Answer: 282

Solution: We want $0 \equiv 10^4 S + 10^3 W + 10^2 E + 10E + T \equiv 8W - 2E + T \pmod{16}$. Let $T = 2U$. Then $U \equiv 4W + E \pmod 8$, where W and E are digits and $U \in \{0, 1, 2, 3, 4\}$.

We now consider cases, being mindful that we want S, W, E, and T to be distinct.

If $W = 0$, then (E, T) can be $(1, 2)$, $(2, 4)$, $(3, 6)$, $(4, 8)$, or $(9, 2)$. For each of these cases, there are 7 possible values for S, hence we have found 35 valid integers so far.

If W is even, but not 0, then (E, T) can be $(1, 2)$, $(2, 4)$, $(3, 6)$, $(4, 8)$, $(8, 0)$, or $(9, 2)$. Mindful that S, W, E, and T must be distinct, we count 18, 12, 18, 12, 21, and 18 valid integers for each of these possibilities, respectively, for a total of 99 more solutions.

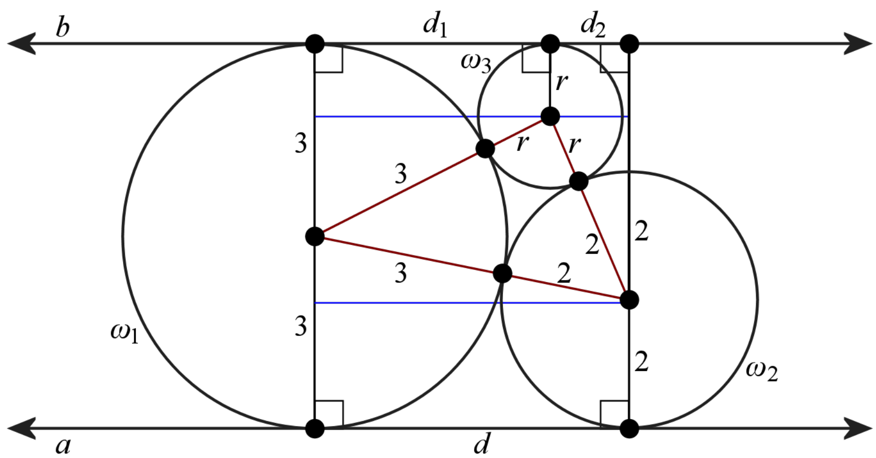
If W is odd, then (E, T) can be $(0, 8)$, $(4, 0)$, $(5, 2)$, $(6, 4)$, or $(7, 6)$. Mindful that S, W, E, and T must be distinct, we count 35, 35, 24, 30, and 24 valid integers for each of these possibilities, respectively, for a total of 148 more solutions.

Adding up these results, the number of valid integers is $35 + 99 + 148 = \boxed{282}$.

Problem 17 Circle ω_1 with radius 3 is inscribed in a strip S having border lines a and b. Circle ω_2 within S with radius 2 is tangent externally to circle ω_1 and is also tangent to line a. Circle ω_3 within S is tangent externally to both circles ω_1 and ω_2, and is also tangent to line b. Compute the radius of circle ω_3. Express your answer as a fraction in simplest form.

Answer: $\dfrac{9}{8}$

Solution:



We compute the distance d indicated in the figure in two different ways. On the one hand, the line parallel to a through the center of ω_2 forms a right triangle with a leg of length d, hence $d = \sqrt{(3+2)^2 - (3-2)^2} = \sqrt{24}$. On the other hand, the line parallel to a through the center of ω_3 creates two right triangles each with a leg parallel to a, and the sum of these leg lengths is also d. Thus,

$$d = \sqrt{24} = d_1 + d_2 = \sqrt{(3+r)^2 - (3-r)^2} + \sqrt{(2+r)^2 - (6-2-r)^2}.$$

Simplifying, we find $\sqrt{24} = \sqrt{12r} + \sqrt{12r - 12}$, or, after dividing by $\sqrt{12}$ and rearranging, $\sqrt{r-1} = \sqrt{2} - \sqrt{r}$. Squaring both sides, we find $r - 1 = 2 - 2\sqrt{2r} + r$, or $2\sqrt{2r} = 3$. Solving for r, we find $r = \boxed{9/8}$.

Problem 18 Let x, y, and z be nonnegative integers that are less than or equal to 100. Suppose that $x + y + z$, $xy + z$, $x + yz$, and xyz are (in some order) four consecutive terms of an arithmetic sequence. Compute the number of such ordered triples (x, y, z).

Answer: 109

Solution: Observe that $AB - A - B = (A-1)(B-1) - 1 \geq 0$ if A, $B > 1$. In other words $A + B \leq AB$ if A and B are both greater than 1.

Now assume that x, y, and z are all greater than 1. By our observation, $x + y + z$ is less than or equal to both $xy + z$ and $x + yz$, and $xy + z$ and

$x + yz$ are both less than or equal to xyz. If the four specified quantities are in arithmetic progression, then the sum of the middle two must equal the sum of the first and last, that is, $(x + y + z) + xyz = (xy + z) + (x + yz)$. Simplifying, this becomes $y + xyz = xy + yz$ or $1 + xz = x + z$, which is impossible since in this case, $xz \geq x + z$.

Next, observe that the set $\{x + y + z, xy + z, x + yz, xyz\}$ is preserved under the swapping of x and z, so we may assume $x \leq z$ to find arithmetic progressions, and then make sure we count triples that can be obtained by swapping x and z when we want to compute the final answer.

If $y = 0$, the four specified quantities becomes $x + z$, z, x, and 0. Since $x \leq z$, these are in arithmetic progression if and only if $z = 2x$ and we get triples $(x, 0, 2x)$ for $0 \leq x \leq 50$.

If $y = 1$, the four specified quantities become $x + z + 1$, $x + z$, $x + z$, and xz, which cannot be in arithmetic progression since evidently two are equal and two are unequal.

Now assume $y > 1$.

If $x = 0$, the specified quantities are $y + z$, z, zy, and 0. If $z = 0$, these evaluate to y, 0, 0, and 0, which cannot be in arithmetic progression since $y > 1$. If $z = 1$, the specified quantities become $y+1$, 1, y, and 0, which form an arithmetic progression only when $y = 2$. If $z > 1$, then $zy > y+z > z > 0$, hence $y = z$ and we must have $zy = z^2 = 3z$, i.e., $z = 3$. We have found the additional triples $(0, 2, 1)$ and $(0, 3, 3)$.

If $x = 1$, the specified quantities are $1 + y + z$, $y + z$, $1 + yz$, and yz. Since all four are integers and $1 + y + z$ and $y + z$ are consecutive, to form an arithmetic progression, they must be consecutive integers. Since $1 + yz$ and yz are also consecutive, we must have either $yz = (y + z + 1) + 1$ or $y + z = (1 + yz) + 1$. In the first case, $yz = (y + z + 1) + 1$, we find $(y - 1)(z - 1) = 3$, which means either $(y, z) = (4, 2)$ or $(y, z) = (2, 4)$. In the second case, $y + z = (1 + yz) + 1$, we find $(y - 1)(z - 1) = -1$, which means either $(y, z) = (2, 0)$ or $(y, z) = (0, 2)$, neither of which respect both $y > 1$ and $x \leq z$. Thus, we only get two more triples: $(1, 4, 2)$ and $(1, 2, 4)$.

If $x > 1$, then we are back in the case where x, y, and z all exceed 1.

To get all the triples, we remember to swap x and z and find the following solutions: $(0, 2, 1)$, $(1, 2, 0)$, $(0, 3, 3)$, $(3, 3, 0)$, $(1, 4, 2)$, $(2, 4, 1)$, $(1, 2, 4)$, $(4, 2, 1)$, $(0, 0, 0)$, and, for each value of x between 1 and 50, inclusive, we get both $(x, 0, 2x)$ and $(2x, 0, x)$, for a total of $\boxed{109}$ solutions.

Problem 19 Up to similarity, there is a unique nondegenerate convex equilateral 13-gon whose internal angles have measures that are multiples of 20 degrees. Find it. Give your answer by listing the degree measures of its 13 *external* angles in clockwise or counterclockwise order. Start your list with the biggest external angle. You don't need to write the degree symbol °.

Answer: 60, 20, 20, 20, 40, 20, 20, 40, 40, 20, 20, 20, 20

Solution: Consider the 18 unit vectors pointing in the directions $20k$ degrees, $k = 0, \ldots, 17$. Because a regular 18-gon has external angles that all measure $360/18 = 20$ degrees, the sum of these 18 vectors is 0. If 13 of these vectors form an equilateral polygon, the other 5 form an equilateral pentagon. Hence, if we can find an equilateral pentagon with external angles that are multiples of 20 degrees, we can take its complement to get the desired 13-gon. We can build such a pentagon by placing an equilateral triangle on a rhombus with external angles of 80 and 100 degrees. The resulting equilateral pentagon has sides pointing in the directions 0, 80, 120, 240, and 260 degrees. The complementary directions, 20, 40, 60, 100, 140, 160, 180, 200, 220, 280, 300, 320, and 340 degrees, form the desired 13-gon. Starting with the biggest external angle, the measures of the external angles (which are the consecutive differences in the list of directions just given) are 60, 20, 20, 20, 40, 20, 20, 40, 40, 20, 20, 20, and 20 degrees.

Problem 20 Compute the value of the sum

$$\sum_{k=1}^{11} \frac{\sin(2^{k+4}\pi/89)}{\sin(2^k\pi/89)}\,.$$

Answer: -2

Solution: We rewrite the equation in terms of exponentials by using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

$$\begin{aligned}
\sum_{k=1}^{11} \frac{\sin(2^{k+4}\pi/89)}{\sin(2^k\pi/89)} &= \sum_{k=1}^{11} \frac{e^{i2^{k+4}\pi/89} - e^{-i2^{k+4}\pi/89}}{e^{i2^k\pi/89} - e^{-i2^k\pi/89}} \\
&= \sum_{k=1}^{11} \frac{(e^{i2^k\pi/89})^{16} - (e^{-i2^k\pi/89})^{16}}{e^{i2^k\pi/89} - e^{-i2^k\pi/89}} \\
&= \sum_{k=1}^{11} \frac{(e^{i2^k\pi/89} - e^{-i2^k\pi/89})(\sum_{j=0}^{15}(e^{i2^k\pi/89})^{15-2j})}{e^{i2^k\pi/89} - e^{-i2^k\pi/89}} \\
&= \sum_{k=1}^{11}\sum_{j=0}^{15}(e^{i2^k\pi/89})^{15-2j}.
\end{aligned}$$

Note that the cyclic subgroup C of $(\mathbb{Z}/89\mathbb{Z})^*$ generated by 2 has 11 elements since $2^{11} = 2048 = 89 \cdot 23 + 1$, i.e., $2^{11} \equiv 1 \pmod{89}$. Therefore, our sum is effectively a sum over cosets of C. Below, we list the cosets of C, one coset per row, and highlight the numbers that are congruent to an odd number between -15 and 15, inclusive, modulo 89:

1	2	4	8	16	32	64	39	**78**	67	45
3	6	12	24	48	**7**	14	28	56	23	46
5	10	20	40	**80**	71	53	17	34	68	47
9	18	36	72	55	21	42	**84**	79	69	49
11	22	44	**88**	87	85	81	73	57	25	50
13	26	52	**15**	30	60	31	62	35	70	51
19	38	**76**	63	37	**74**	59	29	58	27	54
33	66	43	**86**	83	77	65	41	**82**	75	61

Fortunately, we find that every coset has exactly two members that are congruent to an odd number between -15 and 15, inclusive, modulo 89. Thus,

$$\sum_{k=1}^{11}\sum_{j=0}^{15}(e^{i2^k\pi/89})^{15-2j} = 2\sum_{k=1}^{88} e^{2\pi ki/89}.$$

We know that

$$\sum_{k=0}^{88} e^{2\pi ki/89} = 0$$

since the sum represents a regular 89-gon (or deduce this by using the formula for a geometric series). Thus $\sum_{k=1}^{88} e^{2\pi ki/89} = -1$, and the answer is $\boxed{-2}$.

Remark. We wish to point out the related identity

$$\frac{\sin(2nx)}{2\sin x} = \sum_{k=1}^{n} \cos((2k-1)x),$$

which can be seen by computing the center of mass of a uniform circular sector with central angle $2nx$ as the average of the centers of masses of n circular sectors each with central angle $2x$.