Advantage Testing Foundation

The Eleventh Annual Math Prize for Girls

Sunday, October 13, 2019

Test Version A

DIRECTIONS

1. Do not open this test until your proctor instructs you to.
2. Fill out the top of your answer sheet. Your student number is on your name tag. Your test version (A, B, C, or D) is above on this page.
3. You may use pencils, pens, and erasers. You may also use the scratch paper that we provide. You may NOT use calculators, rulers, protractors, compasses, graph paper, or books.
4. Write your final answers on the answer sheet. When a problem specifies a particular form for the answer, write your answer in that form. Write your answers clearly. We will collect only the answer sheet from you.

NOTES

1. This test contains 20 problems. You will have 150 minutes (2.5 hours) to take the test. Your score will be the number of correct answers.
2. Figures are not necessarily drawn to scale.
3. Good luck!

Math Prize for Girls 2019

1. In the USA, standard letter-size paper is 8.5 inches wide and 11 inches long. What is the largest integer that cannot be written as a sum of a whole number (possibly zero) of 8.5's and a whole number (possibly zero) of 11's?

2. Let $a_1, a_2, \ldots, a_{2019}$ be a sequence of real numbers. For every five indices i, j, k, ℓ, and m from 1 through 2019, at least two of the numbers a_i, a_j, a_k, a_ℓ, and a_m have the same absolute value. What is the greatest possible number of distinct real numbers in the given sequence?

3. The degree measures of the six interior angles of a convex hexagon form an arithmetic sequence (not necessarily in cyclic order). The common difference of this arithmetic sequence can be any real number in the open interval $(-D, D)$. Compute the greatest possible value of D.

4. A paper equilateral triangle with area 2019 is folded over a line parallel to one of its sides. What is the greatest possible area of the overlap of folded and unfolded parts of the triangle?

5. Two ants sit at the vertex of the parabola $y = x^2$. One starts walking northeast (i.e., upward along the line $y = x$) and the other starts walking northwest (i.e., upward along the line $y = -x$). Each time they reach the parabola again, they swap directions and continue walking. Both ants walk at the same speed. When the ants meet for the eleventh time (including the time at the origin), their paths will enclose 10 squares. What is the total area of these squares?

6. For each integer from 1 through 2019, Tala calculated the product of its digits. Compute the sum of all 2019 of Tala's products.

7. Mr. Jones teaches algebra. He has a whiteboard with a pre-drawn coordinate grid that runs from -10 to 10 in both the x and y coordinates. Consequently, when he illustrates the graph of a quadratic, he likes to use a quadratic with the following properties:

I The quadratic sends integers to integers.

II The quadratic has distinct integer roots both between -10 and 10, inclusive.

III The vertex of the quadratic has integer x and y coordinates both between -10 and 10, inclusive.

How many quadratics are there with these properties?

8. How many positive integers less than 4000 are not divisible by 2, not divisible by 3, not divisible by 5, and not divisible by 7?

9. Find the least real number K such that for all real numbers x and y, we have $(1+20x^2)(1+19y^2) \geq Kxy$. Express your answer in simplified radical form.

10. A 1×5 rectangle is split into five unit squares (cells) numbered 1 through 5 from left to right. A frog starts at cell 1. Every second it jumps from its current cell to one of the adjacent cells. The frog makes exactly 14 jumps. How many paths can the frog take to finish at cell 5?

11. Twelve 1's and ten -1's are written on a chalkboard. You select 10 of the numbers and compute their product, then add up these products for every way of choosing 10 numbers from the 22 that are written on the chalkboard. What sum do you get?

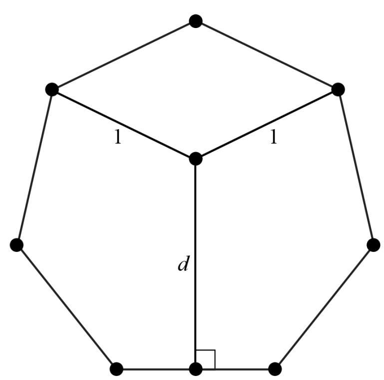
12. Say that a positive integer is MPR (Math Prize Resolvable) if it can be represented as the sum of a 4-digit number MATH and a 5-digit number PRIZE. (Different letters correspond to different digits. The leading digits M and P can't be zero.) Say that a positive integer is MPRUUD (Math Prize Resolvable with Unique Units Digits) if it is MPR and the set of units digits $\{\mathrm{H}, \mathrm{E}\}$ in the definition of MPR can be uniquely identified. Find the smallest positive integer that is MPR but not MPRUUD.

13. Each side of a unit square (side length 1) is also one side of an equilateral triangle that lies in the square. Compute the area of the intersection of (the interiors of) all four triangles. Express your answer in simplified radical form.

14. Devah draws a row of 1000 equally spaced dots on a sheet of paper. She goes through the dots from left to right, one by one, checking if the midpoint between the current dot and some remaining dot to its left is also a remaining dot. If so, she erases the current dot. How many dots does Devah end up erasing?

15. How many ordered pairs (x, y) of real numbers x and y are there such that $-100\pi \le x \le 100\pi$, $-100\pi \le y \le 100\pi$, $x + y = 20.19$, and $\tan x + \tan y = 20.19$?

16. The figure shows a regular heptagon with sides of length 1.



Determine the indicated length d. Express your answer in simplified radical form.

17. Let P be a right prism whose two bases are equilateral triangles with side length 2. The height of P is $2\sqrt{3}$. Let l be the line connecting the centroids of the bases. Remove the solid, keeping only the bases. Rotate one of the bases 180° about l. Let T be the convex hull of the two current triangles. What is the volume of T?

18. How many ordered triples (a, b, c) of integers with $-15 \leq a, b, c \leq 15$ are there such that the three equations $ax + by = c$, $bx + cy = a$, and $cx + ay = b$ correspond to lines that are distinct and concurrent?

19. Consider the base 27 number

$$n = ABCDEFGHIJKLMNOPQRSTUVWXYZ,$$

where each letter has the value of its position in the alphabet. What remainder do you get when you divide n by 100? (The remainder is an integer between 0 and 99, inclusive.)

20. Evaluate the infinite product

$$\prod_{k=2}^{\infty} \left(1 - 4\sin^2 \frac{\pi}{3 \cdot 2^k}\right).$$

Express your answer as a fraction in simplest form.