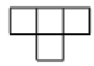
Duke Math Meet 2016

Team Round

1. What is the maximum number of T-shaped polyominos (shown below) that we can put into a 6×6 grid without any overlaps. The blocks can be rotated.



2. In triangle $\triangle ABC$, $\angle A = 30^o$. D is a point on AB such that $CD \perp AB$. E is a point on AC such that $BE \perp AC$. What is the value of $\frac{DE}{BC}$?

3. Given that $f(x)$ is a polynomial such that $2f(x) + f(1-x) = x^2$. Find the sum of squares of the coefficients of $f(x)$.

4. For each positive integer n, there exists a unique positive integer a_n such that $a_n^2 \leq n < (a_n+1)^2$. Given that $n = 15m^2$, where m is a positive integer greater than 1. Find the minimum possible value of $n - a_n^2$.

5. What are the last two digits of $\lfloor(\sqrt{5}+2)^{2016}\rfloor$? Note $\lfloor x \rfloor$ is the largest integer less or equal to x.

6. Let f be a function that satisfies $f(2^a 3^b) = 3a + 5b$. What is the largest value of f over all numbers of the form $n = 2^a 3^b$ where $n \leq 10000$ and a, b are nonnegative integers.

7. Find a multiple of 21 such that it has six more divisors of the form $4m+1$ than divisors of the form $4n+3$ where m, n are integers. You can keep the number in its prime factorization form.

8. Find
$$\sum_{i=0}^{100} \lfloor i^{3/2} \rfloor + \sum_{j=0}^{1000} \lfloor i^{2/3} \rfloor$$
where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

9. Let A, B be two randomly chosen subsets of $\{1, 2, \ldots 10\}$. What is the probability that one of the two subsets contains the other?

10. We want to pick 5-person teams from a total of m people such that:

    1. Any two teams must share exactly one member.
    2. For every pair of people, there is a team in which they are teammates.

    How many teams are there? (Hint: m is determined by these conditions).