The Advantage Testing Foundation

Math Prize for Girls at MIT

2018 Solutions

Problem 1 If x is a real number such that $(x-3)(x-1)(x+1)(x+3)+16 = 116^2$, what is the largest possible value of x?

Answer: 11

Solution: We compute

$$\begin{aligned}(x-3)(x-1)(x+1)(x+3) &= 116^2 - 16 \\ &= 116^2 - 4^2 \\ &= (116+4)(116-4) \\ &= 120 \cdot 112 \\ &= (11^2 - 1^2)(11^2 - 3^2) \\ &= (11+1)(11-1)(11+3)(11-3).\end{aligned}$$

We see that $x = 11$ is a solution. Since all factors in $(x-3)(x-1)(x+1)(x+3)$ are positive and increase with x when $x > 3$, we conclude that the largest solution is $\boxed{11}$.

Problem 2 How many ordered pairs of integers (x, y) satisfy $2|y| \le x \le 40$?

Answer: 841

Solution: Because $2|y| \le 40$, we have $-20 \le y \le 20$. For fixed y, there are $40 - 2|y| + 1$ values of x such that $2|y| \le x \le 40$. We compute

$$\begin{aligned}\sum_{y=-20}^{20} (41 - 2|y|) &= 41^2 - 2 \sum_{y=-20}^{20} |y| \\ &= 41^2 - 4 \sum_{y=1}^{20} y \\ &= 41^2 - 4\frac{20(20+1)}{2} \\ &= \boxed{841}.\end{aligned}$$

Problem 3 Let S be the set of all positive integers from 1 through 1000 that are not perfect squares. What is the length of the longest, non-constant, arithmetic sequence that consists of elements of S?

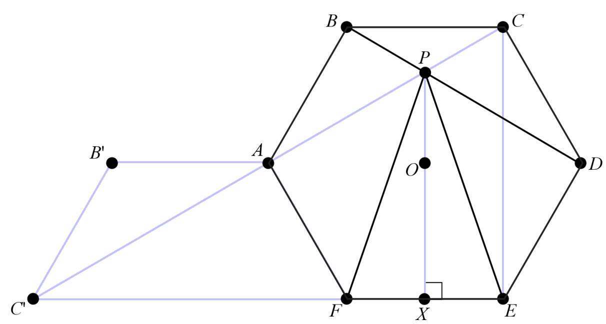
Answer: 333

Solution: Since perfect squares cannot be congruent to 2 mod 3, the subset $\{2, 5, 8, \ldots, 995, 998\}$ of S consisting of numbers that are 2 mod 3 forms an arithmetic sequence devoid of squares and of length $1 + (998 - 2)/3 = 333$. Any arithmetic sequence in S must have an integer common difference. If the common difference exceeds 3, there must be fewer than 251 terms because $1000/n \leq 250$ for $n \geq 4$. The biggest gap between perfect squares in S is $31^2 - 30^2 = 61$ and the biggest gap between perfect squares in S of the same parity is $31^2 - 29^2 = 120$, which show that no arithmetic sequence in S devoid of squares with common difference 1 or 2 can exceed 60 terms. Therefore, the answer is $\boxed{333}$.

Problem 4 Let $ABCDEF$ be a regular hexagon. Let P be the intersection point of $\overline{AC}$ and $\overline{BD}$. Suppose that the area of triangle EFP is 25. What is the area of the hexagon?

Answer: 90

Solution: Refer to the figure below.



Point O is the center of the hexagon and point X is the intersection of $\overleftrightarrow{PO}$ and $\overline{EF}$. By symmetry, $\angle PXF$ is right. Trapezoid $AB'C'F$ by definition is the upper half of a hexagon congruent to $ABCDEF$. The interior angle of a regular hexagon measures $120°$, hence $\angle DEC$, $\angle BAC$, and $\angle B'AC'$ all measure $30°$, as each is the base angle of an isosceles triangle with apex angle $120°$. Therefore $\angle CEF$, $\angle CAF$, and $\angle C'AF$ are all right angles. Since

$\angle CAF$ and $\angle C'AF$ are both right, the points C, A, and C' are collinear, and $\triangle C'XP$ is similar to $\triangle C'EC$. Now, $C'F = 2EF$ and $EF = 2EX$, so $C'X : C'E = 5 : 6$. By similarity, $XP : EC = 5 : 6$. Since $CE = 2OX$, we find $OX : PX = 1/2 : 5/6 = 3/5$. Thus, the area of $\triangle OEF$ is three-fifths the area of $\triangle PEF$, or 15. We conclude that the area of the hexagon is $6 \cdot 15 = \boxed{90}$.

Problem 5 Consider the following system of 7 linear equations with 7 unknowns:

$$\begin{aligned} a+b+c+d+e &= 1 \\ b+c+d+e+f &= 2 \\ c+d+e+f+g &= 3 \\ d+e+f+g+a &= 4 \\ e+f+g+a+b &= 5 \\ f+g+a+b+c &= 6 \\ g+a+b+c+d &= 7. \end{aligned}$$

What is g? Express your answer as a fraction in simplest form.

Answer: $\dfrac{19}{5}$

Solution: Adding up the equations shows that $a+b+c+d+e+f+g = 28/5$. Adding up the third, fifth, and seventh equations shows that

$$2(a+b+c+d+e+f+g) + g = 3 + 5 + 7 = 15.$$

Thus $2(28/5) + g = 15$, from which we find $g = \boxed{19/5}$.

Problem 6 Martha writes down a random mathematical expression consisting of 3 single-digit positive integers with an addition sign "+" or a multiplication sign "×" between each pair of adjacent digits. (For example, her expression could be $4 + 3 \times 3$, with value 13.) Each positive digit is equally likely, each arithmetic sign ("+" or "×") is equally likely, and all choices are independent. What is the expected value (average value) of her expression?

Answer: 50

Solution: Martha can produce four different expression types, each equally likely: $a+b+c$, $a+bc$, $ab+c$, and abc. Therefore, to find the expected value, we can take the average of the expected values for each of these expression types. The expected value of a sum is the sum of the expected values and the expected value of a product of independent random variables is the product of the expected values. Since the expected value of a positive digit picked uniformly at random is 5, we find that the expected value of each expression type is $5+5+5$, $5+5\cdot 5$, $5\cdot 5+5$, and $5\cdot 5\cdot 5$, respectively. Hence, the answer is $(15+30+30+125)/4 = \boxed{50}$.

Problem 7 For every positive integer n, let $T_n = \frac{n(n+1)}{2}$ be the n^{th} triangular number. What is the 2018^{th} smallest positive integer n such that T_n is a multiple of 1000?

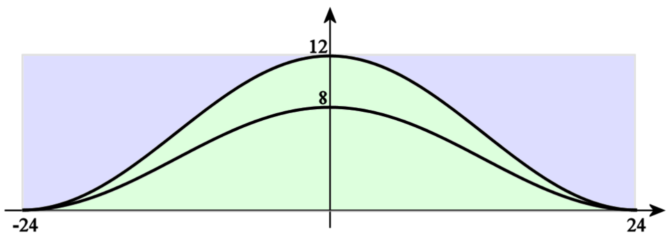
Answer: 1,009,375

Solution: Note that $T_{1000} \equiv 500 \pmod{1000}$ and $T_{2000} \equiv 0 \pmod{1000}$. Thinking of T_n as the sum of the first n positive integers, when we reduce modulo 1000, we see that the sequence of triangular numbers modulo 1000 is periodic with period 2000. Also note that $T_{1999} \equiv T_{2000} - 2000 \equiv 0 \pmod{1000}$. Since $k \equiv -(2000-k) \pmod{1000}$, as you compute successive triangular numbers, you will get the same sequence, modulo 1000, if you start at $T_{1999} \equiv 0 \pmod{1000}$ and work backward. That is, $T_n \equiv T_{1999-n} \pmod{1000}$. Let $0 < n < 1000$. Suppose $T_n = n(n+1)/2 \equiv 0 \pmod{1000}$. Since n and $n+1$ are relatively prime, either n or $n+1$ is divisible by 125. Thus, to find such n, we look at multiples of 125 up to 875 and check if the number one below or one above is divisible by 16. We find $n = 624$ is the only value that works, and hence, the only value strictly between 1000 and 1999 that works is $n = 1999 - 624 = 1375$. Thus T_{2000m} is the $4m$-th triangular number that is a multiple of 1000. The 2018th such number must therefore have index $2000(504) + 1375 = \boxed{1{,}009{,}375}$.

Problem 8 A mustache is created by taking the set of points (x, y) in the xy-coordinate plane that satisfy $4+4\cos(\pi x/24) \le y \le 6+6\cos(\pi x/24)$ and $-24 \le x \le 24$. What is the area of the mustache?

Answer: 96

Solution: Refer to the figure below with green below the bigger cosine curve and purple above.



By symmetry, the green and purple areas are equal and therefore, the area under the cosine curve with the bigger amplitude and above the horizontal axis is $(48 \times 12)/2 = 288$. Similarly, the area under the cosine curve with the smaller amplitude and above the horizontal axis is $(48 \times 8)/2 = 192$. Because $(6 + 6\cos(\pi x/24)) - (4 + 4\cos(\pi x/24)) = 2 + 2\cos(\pi x/24) \geq 0$ for all x, the area between the curves is given by $288 - 192 = \boxed{96}$.

Problem 9 How many 3-term geometric sequences a, b, c are there where a, b, and c are positive integers with $a < b < c$ and $c = 8000$?

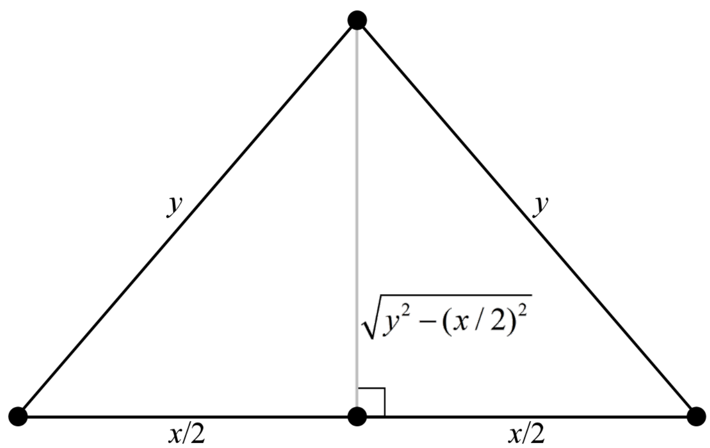
Answer: 39

Solution: Since $a < b < c$ are positive integers, the common ratio must be a positive rational number greater than 1. Let this common ratio be written as p/q for relatively prime integers p and q such that $p > q > 0$. Then $a = 8000(q^2/p^2)$ and $b = 8000(q/p)$. For these to be integers, it necessary and sufficient that p^2 divide evenly into $8000 = 2^6 5^3$. Therefore, p must divide evenly into $2^3 5 = 40$. Since any fraction q/p where p divides evenly into 40 can be uniquely expressed as a fraction $x/40$, where x is an integer, and since fractions of the form $x/40$ are different for different values of x, we conclude that for each value of x from 1 to 39, inclusive, we obtain a common ratio that yields a desired geometric sequence. Thus, the answer is $\boxed{39}$.

Problem 10 Let T_1 be an isosceles triangle with sides of length 8, 11, and 11. Let T_2 be an isosceles triangle with sides of length b, 1, and 1. Suppose that the radius of the incircle of T_1 divided by the radius of the circumcircle of T_1 is equal to the radius of the incircle of T_2 divided by the radius of the circumcircle of T_2. Determine the largest possible value of b. Express your answer as a fraction in simplest form.

Answer: $\dfrac{14}{11}$

Solution: Refer to the figure below.



Let T be an isosceles triangle with sides of length x, y, and y. By the law of sines, the radius of the circumcircle of T is equal to y divided by the sine of a base angle: $\frac{1}{2}\frac{y^2}{\sqrt{y^2-x^2/4}}$. The radius of the incircle of T is the ratio of the area of T to its semiperimeter: $\frac{x\sqrt{y^2-x^2/4}}{x+2y}$. Therefore, the ratio of the radius of the incircle of T to the radius of its circumcircle is given by

$$\frac{2x(y^2-x^2/4)}{y^2(x+2y)} = \frac{x(y-x/2)}{y^2} = (x/y)-(x/y)^2/2.$$

This shows that the ratio x/y is a root of a quadratic of the form $z^2/2-z+c$, where c is the ratio of the radius of the triangle's incircle to that of its circumcircle.

By Vieta's formulas, we conclude that b is the larger of $8/11$ or $2-8/11 = 14/11$, hence the answer is $\boxed{14/11}$.

Problem 11 Maryam has a fair tetrahedral die, with the four faces of the die labeled 1 through 4. At each step, she rolls the die and records which number is on the bottom face. She stops when the current number is greater than or equal to the previous number. (In particular, she takes at least two steps.) What is the expected number (average number) of steps that she takes? Express your answer as a fraction in simplest form.

Answer: $\dfrac{625}{256}$

Solution: Let X be the random variable that gives the number of steps Maryam takes. Then $E(X) = \sum_{k=0}^{\infty} kP(X=k) = \sum_{k=0}^{\infty} P(X \geq k)$. Now

$P(X \geq k)$ is the probability that Maryam's game lasts at least k rolls. Since the game must last more than 1 roll, we know $P(X \geq 0) = 1$, so assume $k > 0$. For the game to last at least k rolls, Maryam's first $k-1$ rolls must be a strictly decreasing sequence. For every subset of $k-1$ distinct die outcomes, there is a unique way to arrange them in decreasing order. Thus, there are $\binom{4}{k-1}$ ways to get $k-1$ decreasing outcomes in a row. Since there are 4^{k-1} possible ways to roll $k-1$ tetrahedral dice, we have $P(X \geq k) = \binom{4}{k-1}/4^{k-1}$. Therefore,

$$E(X) = \sum_{k=0}^{\infty} \binom{4}{k-1} \left(\frac{1}{4}\right)^{k-1} = \left(1 + \frac{1}{4}\right)^4 = \boxed{\frac{625}{256}},$$

where we have used the convention that $\binom{n}{k} = 0$ if $k < 0$ or $k > n$.

Problem 12 You own a calculator that computes exactly. It has all the standard buttons, including a button that replaces the number currently displayed with its arctangent, and a button that replaces whatever is currently displayed with its cosine. You turn on the calculator and it reads 0. You create a sequence by alternately clicking on the arctangent button and the cosine button. (The calculator is in radian mode.) Let a_n be the value displayed after you've pressed the cosine button for the nth time. What is $\prod_{k=1}^{11} a_k$? Express your answer as a fraction in simplest form.

Answer: $\frac{1}{12}$

Solution: Since the arctangent of 0 is 0, we may assume that we repeatedly apply the function $g(x) = \cos(\arctan(x))$. Then $a_n = g^{(n)}(0)$.

Let $F_1 = F_2 = 1$ and define F_n recursively by the rule $F_{n+1} = F_n + F_{n-1}$ for $n > 1$, i.e. F_n is the Fibonacci sequence. For $n > 0$, we compute

$$g\left(\sqrt{\frac{F_n}{F_{n+1}}}\right) = \frac{\sqrt{F_{n+1}}}{\sqrt{F_n + F_{n+1}}} = \frac{\sqrt{F_{n+1}}}{\sqrt{F_{n+2}}} = \sqrt{\frac{F_{n+1}}{F_{n+2}}}.$$

Since $g(0) = 1 = \sqrt{\frac{F_1}{F_2}}$, we see, by induction, that $a_n = \sqrt{\frac{F_n}{F_{n+1}}}$. Thus

$$\prod_{k=1}^{11} a_k = \prod_{k=1}^{11} \sqrt{\frac{F_n}{F_{n+1}}} = \sqrt{\frac{F_1}{F_{12}}} = \sqrt{\frac{1}{144}} = \boxed{\frac{1}{12}}.$$

Problem 13 A circle overlaps an equilateral triangle of side length $100\sqrt{3}$. The three chords in the circle formed by the three sides of the triangle have lengths 6, 36, and 60, respectively. What is the area of the circle? Express your answer in terms of π.

Answer: 2925π

Solution: Let r be the radius of the circle. The distances of the center of the circle from the three sides are $\sqrt{r^2-3^2}$, $\sqrt{r^2-18^2}$, and $\sqrt{r^2-30^2}$. In an equilateral triangle, the lengths of the 3 perpendiculars from an interior point to the sides sum to the height of the triangle. Thus,

$$\sqrt{r^2-3^2}+\sqrt{r^2-18^2}+\sqrt{r^2-30^2}=150.$$

To solve, let's attempt to find a value of r which makes each term in a radical a perfect square. So suppose there exists integers a, b, and c where

$$\begin{aligned} a^2 &= r^2-3^2 \\ b^2 &= r^2-18^2 \\ c^2 &= r^2-30^2. \end{aligned}$$

Then $a^2-b^2=18^2-3^2=21\cdot 15=3^2(5)(7)$ and $a^2-c^2=30^2-3^2=33\cdot 27=3^4\cdot 11$. From the first of these equations, we find the possible values of (a,b) to be $(158,157)$, $(54,51)$, $(34,29)$, $(26,19)$, $(22,13)$, and $(18,3)$. From the second of these equations, we find the possible values of (a,c) to be $(446,445)$, $(150,147)$, $(54,45)$, $(46,35)$, and $(30,3)$. Fortunately, there are two consistent pairs, namely $(a,b)=(54,51)$ and $(a,c)=(54,45)$. We conclude that $r^2=54^2+3^2=51^2+18^2=45^2+30^2=2925$. Therefore, the answer is $\boxed{2925\pi}$.

Problem 14 Let $f(x)$ be the polynomial $\prod_{k=1}^{50}\big(x-(2k-1)\big)$. Let c be the coefficient of x^{48} in $f(x)$. When c is divided by 101, what is the remainder? (The remainder is an integer between 0 and 100.)

Answer: 60

Solution: Let $g(x)=\prod_{k=1}^{50}(x-2k)$. Then $f(x)\equiv g(-x) \pmod{101}$. Therefore, if we write $f(x)=x^{50}+bx^{49}+cx^{48}+\ldots$, then $g(x)\equiv x^{50}-bx^{49}+cx^{48}-\ldots \pmod{101}$. Since 101 is prime, by Fermat's little theorem, we have $f(x)g(x)\equiv x^{101-1}-1 \pmod{101}$. Therefore, the coefficient of x^{98} in $f(x)g(x)$, which is $2c-b^2$, must be congruent to 0 modulo 101. Modulo 101, we know that b is the sum of the even residues, which is equivalent to $50(51)$. We compute

$$c \equiv 50^2 51^2/2 \equiv 50^2 51^3 \equiv -50^5 \equiv -5^5(10) \equiv -19(5)(10) \equiv 60 \pmod{101}.$$

Therefore, the answer is $\boxed{60}$.

Problem 15 In the xy-coordinate plane, the x-axis and the line $y = x$ are mirrors. If you shoot a laser beam from the point $(126, 21)$ toward a point on the positive x-axis, there are 3 places you can aim at where the beam will bounce off the mirrors and eventually return to $(126, 21)$. They are $(126, 0)$, $(105, 0)$, and a third point $(d, 0)$. What is d? (Recall that when light bounces off a mirror, the angle of incidence has the same measure as the angle of reflection.)

Answer: 111

Solution: Given a triangle T (without a right angle), its altitude triangle (sometimes called its orthic triangle) is the triangle whose vertices are the feet of the altitudes of T. We will use the fact that the altitude triangle of an acute triangle T is a light path in T. We will apply this fact to a triangle $T = \triangle ABC$ with A at the origin, B on the positive x-axis, and C on the line $y = x$. We will make the line segment connecting $A = (0, 0)$ to $(126, 21)$ an altitude of T. Since that altitude has slope $1/6$, the slope of side $\overline{BC}$ must be -6. The x-coordinate of C is the solution to the equation $x - 21 = -6(x - 126)$. We solve this and find that $x = 111$. Since the altitude from C is vertical, we conclude that the answer is $\boxed{111}$.

Problem 16 Define a function f on the unit interval $0 \le x \le 1$ by the rule

$$f(x) = \begin{cases} 1 - 3x & \text{if } 0 \le x < 1/3\,; \\ 3x - 1 & \text{if } 1/3 \le x < 2/3\,; \\ 3 - 3x & \text{if } 2/3 \le x \le 1\,. \end{cases}$$

Determine $f^{(2018)}(1/730)$. Express your answer as a fraction in simplest form. Recall that $f^{(n)}$ denotes the nth iterate of f; for example, $f^{(3)}(1/730) = f(f(f(1/730)))$.

Answer: $\dfrac{9}{730}$

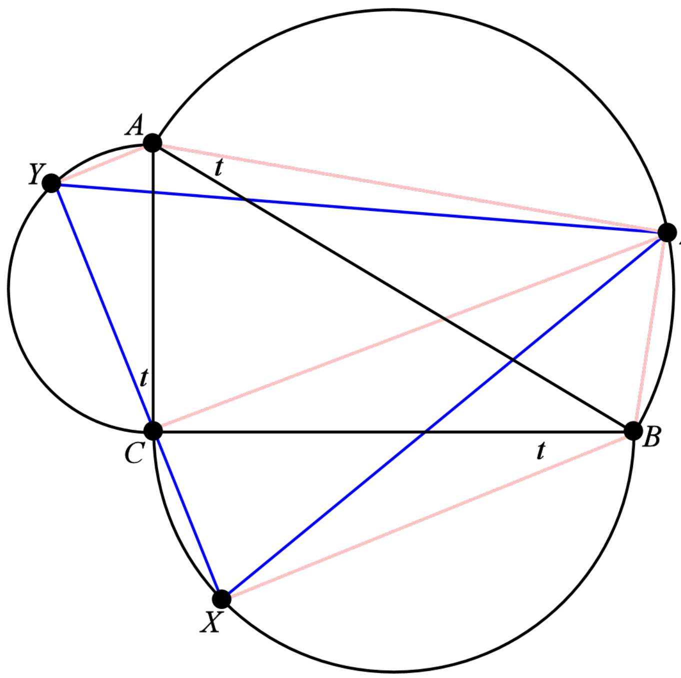
Solution: The effect of applying f on the ternary digits of a number is to shift the digits left by 1 place, eliminate the units digit, and, if the units digits was 0 or 2, complement the number (i.e., swap the digits 0 and 2 throughout). Since $730 = 3^6 + 1$, the fraction $1/730 = (3^6 - 1)/(3^{12} - 1)$ has the ternary expansion $0.\overline{000000222222}$. Since this ternary number has period 12 and an even number of 0's and 2's within a period, we see that $f^{(12)}(1/730) = 1/730$. Therefore,

$$f^{(2018)}(1/730) = f^{(2)}(1/730) = f(727/730) = \boxed{9/730}.$$

Problem 17 Let ABC be a triangle with $AB = 5$, $BC = 4$, and $CA = 3$. On each side of ABC, externally erect a semicircle whose diameter is the corresponding side. Let X be on the semicircular arc erected on side $\overline{BC}$ such that $\angle CBX$ has measure 15°. Let Y be on the semicircular arc erected on side $\overline{CA}$ such that $\angle ACY$ has measure 15°. Similarly, let Z be on the semicircular arc erected on side $\overline{AB}$ such that $\angle BAZ$ has measure 15°. What is the area of triangle XYZ? Express your answer as a fraction in simplest form.

Answer: $\dfrac{73}{8}$

Solution: Let $t = 15^\circ$ and refer to the figure below.



Note that $\triangle ABC$ is right with right angle at C. Hence, C is on the circle that extends the semicircle whose diameter is $\overline{AB}$. Therefore, $m\angle ZCB = m\angle ZAB = t$. Since $\angle BCX$ is complementary to $\angle CBX$, we see that $\angle ZCX$ is right. Also note that $m\angle YCX = m\angle YCA + m\angle ACB + m\angle BCX = t + 90^\circ + (90^\circ - t) = 180^\circ$, and, hence, Y, C, and X are collinear. Since $\overline{ZC}$, $\overline{AY}$, and $\overline{BX}$ are all perpendicular to $\overline{XY}$, they are parallel to each other. By Cavalieri's principle (applied twice), $\triangle CZX$ has the same area as quadrilateral $AZBC$, which, in turn, is the sum of the areas of right triangles ABC and ABZ. The area of $\triangle ABC$ is $\frac{1}{2}3 \cdot 4 = 6$. The area of $\triangle ABZ$ is $\frac{1}{2}(5\cos t)(5\sin t) = \frac{25}{4}\sin(2t) = \frac{25}{4}\sin(30^\circ) = \frac{25}{8}$. Therefore, the answer is $6 + \dfrac{25}{8} = \boxed{\dfrac{73}{8}}$.

Problem 18 Evaluate the expression

$$\left|\prod_{k=0}^{15}\left(1 + e^{2\pi i k^2/31}\right)\right|.$$

Answer: 2

Solution: Let $p = 31$. Since $p \equiv 3 \pmod 4$, we know -1 is not a square modulo p. Define

$$f(x) = \prod_{k=1}^{(p-1)/2} \left(x - e^{2\pi i k^2/p} \right),$$

i.e., $f(x)$ is a polynomial whose roots are the roots of unity corresponding to nonzero quadratic residues modulo p. Since complex conjugation of a root of unity $e^{i\theta}$ is equivalent to multiplying its exponent by -1, and since -1 is not a square modulo p, the roots of the polynomial $\overline{f(\overline{x})}$ are the roots of unity corresponding to quadratic nonresidues, modulo p. Therefore, $x^p - 1 = (x-1)f(x)\overline{f(\overline{x})}$. Substituting -1 for x and taking absolute values, we find $|f(-1)| = 1$. The product in the problem statement is $|(-1-1)f(-1)| = \boxed{2}$.

Problem 19 Consider the sum

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{2k-1}}.$$

Determine $\lfloor S_{4901} \rfloor$. Recall that if x is a real number, then $\lfloor x \rfloor$ (the *floor* of x) is the greatest integer that is less than or equal to x.

Answer: 98

Solution: Note that

$$\sqrt{k+2} - \sqrt{k} = (\sqrt{k+2} - \sqrt{k}) \frac{\sqrt{k+2} + \sqrt{k}}{\sqrt{k+2} + \sqrt{k}} = \frac{2}{\sqrt{k+2} + \sqrt{k}}.$$

Also observe that

$$\frac{1}{\sqrt{k+2}} = \frac{2}{2\sqrt{k+2}} < \frac{2}{\sqrt{k+2} + \sqrt{k}} < \frac{2}{\sqrt{k} + \sqrt{k}} = \frac{1}{\sqrt{k}}.$$

Therefore,

$$\frac{1}{\sqrt{k+2}} < \sqrt{k+2} - \sqrt{k} < \frac{1}{\sqrt{k}}. \qquad (*)$$

Summing the left side and middle of $(*)$ over odd numbers k from 1 to $2n-3$, we find that

$$S_n - 1 < \sqrt{2n-1} - 1.$$

Summing the middle and right side of $(*)$ over odd numbers k from 1 to $2n-1$, we find that

$$\sqrt{2n+1} - 1 < S_n.$$

Thus,

$$\sqrt{2n+1} - 1 < S_n < \sqrt{2n-1}.$$

Finally, note that $\sqrt{2(4901)-1} = \sqrt{9801} = 99$. Thus, $98 < S_{4901} < 99$, and the answer is $\boxed{98}$.

Problem 20 A *smooth number* is a positive integer of the form $2^m 3^n$, where m and n are nonnegative integers. Let S be the set of all triples (a, b, c) where a, b, and c are smooth numbers such that $\gcd(a, b)$, $\gcd(b, c)$, and $\gcd(c, a)$ are all distinct. Evaluate the infinite sum $\sum_{(a,b,c)\in S} \frac{1}{abc}$. Express your answer as a fraction in simplest form. Recall that $\gcd(x, y)$ is the greatest common divisor of x and y.

Answer: $\dfrac{162}{91}$

Solution: Consider the triple $s = (2^a 3^x, 2^b 3^y, 2^c 3^z)$. We claim that $s \in S$ if and only if one of the following six mutually exclusive cases holds:

$$\begin{aligned}
a < \min(b, c) &\text{ and } y < \min(z, x) \\
a < \min(b, c) &\text{ and } z < \min(x, y) \\
b < \min(c, a) &\text{ and } z < \min(x, y) \\
b < \min(c, a) &\text{ and } x < \min(y, z) \\
c < \min(a, b) &\text{ and } x < \min(y, z) \\
c < \min(a, b) &\text{ and } y < \min(z, x)
\end{aligned} \tag{1}$$

To see this, suppose $a < \min(b, c)$ and $y < \min(z, x)$. Then

$$\begin{aligned}
\gcd(2^a 3^x, 2^b 3^y) &= 2^a 3^y, \\
\gcd(2^b 3^y, 2^c 3^z) &= 2^{\min(b,c)} 3^y, \\
\text{and } \gcd(2^c 3^z, 2^a 3^x) &= 2^a 3^{\min(x,z)}.
\end{aligned}$$

Since $a < \min(b, c)$, the first is less than the second, and the second is not equal to the third. Since $y < \min(x, z)$, the first is less than the third. Therefore, these three gcd's are distinct and $s \in S$. Similar reasoning applies to the other 5 cases.

Conversely, suppose $s \in S$.

We first show that we cannot have $a = b = c$, for suppose otherwise. If $x = \min(x, y, z)$, then $\gcd(2^a 3^x, 2^b 3^y) = \gcd(2^a 3^x, 2^c 3^z) = 2^a 3^x$. A similar argument shows that no matter which of x, y, and z is minimal, we would get a pair of gcd's that are equal. A similar argument shows that x, y, and z cannot all be equal to each other.

Next we show that we cannot have two of a, b, and c equal to each other *and* equal to $\min(a, b, c)$. For suppose $a = b = \min(a, b, c)$. If $x = \min(x, y, z)$, then $\gcd(2^a 3^x, 2^b 3^y) = \gcd(2^a 3^x, 2^c 3^z) = 2^a 3^x$, and a similar computation eliminates the cases where $y = \min(x, y, z)$ and $z = \min(x, y, z)$. Similar reasoning eliminates the cases where $b = c = \min(a, b, c)$ and $c = a = \min(a, b, c)$. And a similar argument shows that we also cannot have two of x, y, and z equal to each other *and* equal to $\min(x, y, z)$.

Next, we claim that we cannot have $a < \min(b, c)$ and $x < \min(y, z)$, for in this case, we would have $\gcd(2^a 3^x, 2^b 3^y) = \gcd(2^a 3^x, 2^c 3^z) = 2^a 3^x$. Similarly, we cannot have both $b < \min(c, a)$ and $y < \min(z, x)$, nor can we have both $c < \min(a, b)$ and $z < \min(x, y)$.

Therefore s must fall into one of the 6 cases (1).

The 6 cases (1) are symmetric, so each contributes an equal amount to the sum. We compute the contribution of one case then multiply the result by 6 to get the final answer.

$$\begin{aligned}
\sum_{\substack{(2^a 3^x, 2^b 3^y, 2^c 3^z) \in S \\ a < \min(b,c) \\ y < \min(z,x)}} \frac{1}{2^{a+b+c} 3^{x+y+z}} &= \sum_{a=0}^{\infty} \sum_{b=a+1}^{\infty} \sum_{c=a+1}^{\infty} \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} \sum_{z=y+1}^{\infty} \frac{1}{2^{a+b+c} 3^{x+y+z}} \\
&= \sum_{a=0}^{\infty} \sum_{b=a+1}^{\infty} \sum_{c=a+1}^{\infty} \sum_{y=0}^{\infty} \frac{1}{2^{a+b+c+2} 3^{3y}} \\
&= \frac{27}{4 \cdot 26} \sum_{a=0}^{\infty} \sum_{b=a+1}^{\infty} \sum_{c=a+1}^{\infty} \frac{1}{2^{a+b+c}} \\
&= \frac{27}{4 \cdot 26} \sum_{a=0}^{\infty} \frac{1}{2^{3a}} \\
&= \frac{27}{4 \cdot 26} \frac{8}{7} = \frac{27}{91}.
\end{aligned}$$

The final answer is $6 \cdot \dfrac{27}{91} = \boxed{\dfrac{162}{91}}$.