Duke Math Meet
Team Round Solutions

January 7, 2018

1. What is the maximum possible value of m such that there exist m integers $a_1, a_2, ..., a_m$ where all the decimal representations of $a_1!, a_2!, ..., a_m!$ end with the same amount of zeros?

   Solution. Answer: $\boxed{5}$

   The number of zeros in the decimal representation of $n!$ is equal to

   $$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + ...$$

   Hence, $5k!, (5k+1)!, (5k+2)!, (5k+3)!$, and $(5k+4)!$ end with the same amount of zeros for all k.

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2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

   $$f(x) + f(y^2) = f(x^2 + y), \text{ for all } x, y \in \mathbb{R}.$$

   Find the sum of all possible $f(-2017)$.

   Solution. Answer: $\boxed{0}$

   Substitute $x = y = 0$ to get $f(0) = 0$.

   Substitute $y = 0$ to get $f(x) = f(x^2)$, which implies $f(x) = f(-x)$.

   We have $f(x^2 + y) = f(x) + f(y^2) = f(x^2) + f(y) = f(y^2 + x)$.

   Substitute $y = -x^2$ to obtain $0 = f(0) = f(x^4 + x)$.

   Observe that $g(x) = x^4 + x$ is a continuous function, $g(0) = 0$, and $g(x)$ approaches infinity when x approaches infinity. It follows that for all positive real y, there exist x such that $g(x) = y$ which implies $f(0) = f(g(x)) = f(y) = f(-y)$. Therefore, $f(x) = 0$ for all x.

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3. What is the sum of prime factors of 1000027?

   Solution. Answer: $\boxed{202}$

   $$1000027 = 100^3 + 3^3 = 103(100^2 - 100.3 + 3^2) = 103.(103^2 - 100.3.3) = 103.73.133 = 103.73.7.19$$

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4. Let
$$\frac{1}{2!}+\frac{2}{3!}+...+\frac{2016}{2017!}=\frac{n}{m},$$
where n, m are relatively prime. Find $(m-n)$.

Solution. Answer: $\boxed{1}$

$$\frac{1}{2!}+\frac{2}{3!}+...+\frac{2016}{2017!}=(1-\frac{1}{2!})+(\frac{1}{2!}-\frac{1}{3!})+...+(\frac{1}{2016!}-\frac{1}{2017!})=1-\frac{1}{2017!}=\frac{2017!-1}{2017!}$$

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5. Determine the number of ordered pairs of real numbers (x, y) such that
$$\sqrt[3]{3-x^3-y^3}=\sqrt{2-x^2-y^2}.$$

Solution. Answer: $\boxed{0}$

Set $z=\sqrt[3]{3-x^3-y^3}=\sqrt{2-x^2-y^2}$, we have
$$x^2+y^2+z^2=2, \quad x^3+y^3+z^3=3.$$
Since, $3^2>2^3$, we must have
$$(x^3+y^3+z^3)^2>(z^2+y^2+z^2)^3$$

$$2x^3y^3+2y^3z^3+2z^3x^3>x^2y^2(x^2+y^2)+y^2z^2(y^2+z^2)+z^2x^2(x^2+z^2)$$

$$0>x^2y^2(x-y)^2+y^2z^2(y-z)^2+z^2x^2(z-x)^2 \quad \text{(Contradiction)}.$$

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6. Triangle $\triangle ABC$ has $\angle B=120°$, $AB=1$. Find the largest real number x such that $CA-CB>x$ for all possible triangles $\triangle ABC$.

Solution. Answer: $\boxed{1/2}$

Let AH be perpendicular to BC at H. Since $\angle HBA=60°$, $HB=1/2$ and $AH=\sqrt{3}/2$.

We have $3/4=AH^2=AC^2-CH^2=(CA-CB-1/2)(CA+CB+1/2)$.

This implies $CA-CB>1/2$ and approaches $1/2$ as $CA+CB$ approaches infinity.

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7. Jung and Remy are playing a game with an unfair coin. The coin has a probability of p where its outcome is heads. Each round, Jung and Remy take turns to flip the coin, starting with Jung in round 1. Whoever gets heads first wins the game. Given that Jung has the probability of $\frac{8}{15}$, what is the value of p?

Solution. Answer: $\boxed{1/8}$

If Jung flips heads in round 1, he wins and the game ends. If Jung flips tails in round 1 and Remy flips heads in round 2, Remy wins. If Jung and Remy both flip tails in the first two rounds, the game starts over and Jung has probability of Jung's winning is again $\frac{8}{15}$. Therefore,

$$P(\text{Jung wins}) = \frac{8}{15} = p + (1-p)(1-p)\frac{8}{15}.$$

Solving for p, ew get that $p = 0$ or $p = \frac{1}{8}$, which leads us to our more reasonable answer of $\boxed{\frac{1}{8}}$. ■

8. Consider a circle with 7 equally spaced points marked on it. Each point is 1 unit distance away from its neighbors and labelled 0,1,2,...,6 in that order counterclockwise. Feng is to jump around the circle, starting at the point 0 and making six jumps counterclockwise with distinct lengths $a_1, a_2, ..., a_6$ in a way such that he will land on all other six nonzero points afterwards. Let s denote the maximum value of a_i. What is the minimum possible value of s?

   Solution. Answer: $\boxed{8}$

   First, s cannot be 7 because a jump of 7 around the circle lands in the same spot. When $s = 8$, one can pick $(1, 2, 3, 5, 8, 4)$ the be the ordering of the jumps.

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9. Justin has a $4 \times 4 \times 4$ colorless cube that is made of 64 unit-cubes. He then colors m unit-cubes such that none of them belong to the same column or row of the original cube. What is the largest possible value of m?

   Solution. Answer: $\boxed{16}$

   Let mark 64 unit-cubes by coordinates $(i, j, k)'s, 1 \leq i, j, k \leq 4$ so that two unit-cubes belong to the same row/column iff they have 2 equal coordinates. According to the Pigeonhole principle, if 17 unit-cubes are colored, 2 of them must have the same first two coordinates.

   On the other hand, a coloring of 16 unit-cubes that satisfies the requirement is

   {(1,1,1), (1,2,2), (1,3,3), (1,4,4); (2,1,2), (2,2,3), (2,3,4), (2,4,1); (3,1,3), (3,2,4), (3,3,1), (3,4,2); (4,1,4), (4,2,1), (4,3,2), (4,4,3)}. ■

10. Yikai wants to know Liang's secret code which is a 6-digit integer x. Furthermore, let $d(n)$ denote the digital sum of a positive integer n. For instance, $d(14) = 5$ and $d(3) = 3$. It is given that

    $$x + d(x) + d(d(x)) + d(d(d(x))) = 999868.$$

    Please find x

    Solution. Answer: $\boxed{999820}$ ■