

Duke Math Meet 2017

Individual Round

October 31, 2017

1. How many subsets of $\{D, U, K, E\}$ have an odd number of elements?
Answer:
2. Find the coefficient of x^{12} in $(1 + x^2 + x^4 + \dots + x^{28})(1 + x + x^2 + \dots + x^{14})^2$.
Answer:
3. How many 4-digit numbers have their digits in non-decreasing order from left to right?
Answer:
4. A dodecahedron (a polyhedron with 12 faces, each a regular pentagon) is projected orthogonally onto a plane parallel to one of its faces to form a polygon. Find the measure (in degrees) of the largest interior angle of this polygon.
Answer:
5. Justin is back with a 6×6 grid made of 36 colorless squares. Dr. Kraines wants him to color some squares such that
 - Each row and column of the grid must have at least one colored square
 - For each colored square, there must be another colored square on the same row or column

What is the minimum number of squares that Justin will have to color?

Answer:

6. Inside a circle C , we have three equal circles C_1, C_2, C_3 , which are pairwise externally tangent to each other and all internally tangent to C . What is the ratio of the area of C_1 to the area of C ?
Answer:
7. There are 3 different paths between the Duke Chapel and the Physics building. 6 students are heading towards the Physics building for a class, so they split into 3 pairs and each pair takes a separate path from the Chapel. After class, they again split into 3 pairs and take separate paths back. Find the number of possible scenarios where each student's companion on the way there is different from their companion on the way back.
Answer:
8. Let a_n be a sequence that satisfies the recurrence relation

$$a_n a_{n+2} = \frac{\cos(3a_{n+1})}{\cos(a_{n+1})[2\cos(2a_{n+1}) - 1]} a_{n+1}$$

with $a_1 = 2$ and $a_2 = 3$. Find the value of 2018 a_{2017} .

Answer:

9. Let $f(x)$ be a polynomial with minimum degree, integer coefficients, and leading coefficient of 1 that satisfies $f(\sqrt{7} + \sqrt{13}) = 0$. What is the value of $f(10)$?

Answer:

10. 1024 Duke students, indexed 1 to 1024, are having a chat. For each $1 \leq i \leq 1023$, student i claims that student $2^{\lfloor \log_2 i \rfloor + 1}$ has a girlfriend. ($\lfloor x \rfloor$ is the greatest integer less than or equal to x .) Given that exactly 201 people are lying, find the index of the 61st liar (ordered by index from smallest to largest).

Answer: