Duke Math Meet 2016

Team Solutions

1. It is easy to show that $\boxed{8}$ is possible. First each tile has area 4 and total area is 36 so we can have at most 9 tiles. We will show that 9 is not possible. Color the 6×6 grid in a checkboard pattern. Then there are an 18 white squares. Each tile will cover either 1 or 3 white squares. Hence 9 tiles will cover an odd number of white squares. This isn't possible so the maximum is 8.

2. $\angle BEC = 90^o$ and $\angle CDB = 90^o$. So $BECD$ is a cyclic quadrilateral. Let F be the intersection of BD and CE. Then $\triangle DEF$ is similar to BCF. Hence $\frac{DE}{BC} = \frac{FD}{BF}$ but triangle BDF is a 30-60-90 right triangle so $\frac{FD}{BF} = \boxed{\frac{\sqrt{3}}{2}}$.

3. So we have $2f(x)+f(1-x) = x^2$ and $2f(1-x)+f(x) = (1-x)^2$ (we substitute $1-x$ for x). We can solve this as a system of linear equations. If we multiply the first equation by 2 and then subtract the second, we see that $3f(x) = 2x^2 - (1-x)^2 = x^2 - 2x - 1$. Hence the sum of the coefficients is $\frac{1}{9}(1+4+1) = \boxed{\frac{2}{3}}$.

4. We want to find the minimum integer k of the form $15m^2 - a^2$ where $15m^2 < a^2 - 1$. Checking $k = 0, \ldots, 5$, all will not satisfy through modulo $3, 2,$ and 5. $15m^2 - 6 = a^2$. So $a = 3b$ so we have $5m^2 - 2 = 3b^2$. Modulo 5, we see that $b \equiv 1, 4 \pmod 5$. Trying possibilities, we see that $b = 9, m = 7$ works. So the answer is $\boxed{6}$.

5. Expanding $(\sqrt{5}+2)^{2016}+(\sqrt{5}-2)^{2016}$, we see that it is equal to an integer. In addition $\sqrt{5}-2 < 1$ so any power of it is less than 1. So $\lfloor(\sqrt{5}+2)^{2016}\rfloor = (\sqrt{5}+2)^{2016}+(\sqrt{5}-2)^{2016}-1$. Since we only need the last two digits, we can consider the expression mod 100. $(\sqrt{5}+2)^{2016}+(\sqrt{5}-2)^{2016} = 2(5)^{1008} + 2\binom{2016}{2}(5)^{1007}2^2 + \ldots + \binom{2016}{2}2(5)2^{2014} + 2(2)^{2016}$. Note that most terms are divisible by 100 so we can ignore them. So we have $2(5)^{1008} + +2^{2017}$. 2^{2017} repeats every $\phi(25) = 20$ mod 25 so we have $2^{2017} \equiv 2^{17} \equiv 2^{-3} \equiv -3 \pmod 25$. So $2^{2017} \equiv 72 \pmod 100$. $2(5)^{1008} + 2^{2017} \equiv 50 + 72 \equiv 22 \pmod 100$. Hence $22 - 1 = \boxed{21}$

6. Suppose $f(2^a3^b)$ is the maximum over the given range with 2^a3^b smallest. Since $f(2^{a-5}3^{b+3}) = f(2^a3^b)$ but $2^{a-5}3^{b+3} = \frac{27}{32}2^a3^b$. Then we see that $a < 5$ otherwise we contradict our minimality of 2^a3^b. For a fixed a, we just want to pick the largest b such that $2^a3^b \leq 10000$. For powers of 3, we have $1, 3, 9, 27, 243, 729, 2187, 6561$. So when $a = 0$, we get $b = 8$. $a = 1 \implies b = 7$, $a = 2 \implies b = 7$, $a = 3 \implies b = 6$, $a = 4 \implies b = 5$. Calculating $3a + 5b$ we see that the maximum is when $a = 2, b = 7$ so $3(2) + 7(5) = \boxed{41}$.

7. Considering only $4n+3$ primes. Suppose x is any odd number and p a $4n+3$ prime. Then $x, px, p^2x, \ldots, p^{2n-1}x$ contains the same number of $4n+1$ numbers as $4n+3$. So we see that every power of a $4n+3$ prime must be even. In addition, we can see that

when all the $4n+3$ primes are raised to an even power that is has precisely 1 more $4n+1$ divisor than $4n+3$. Now consider the $4n+1$ prime factors. Let x be the product of all $4n+1$ prime factors. Since multiplying by $4n+1$ doesn't change if the number is $4n+1$ or $4n+3$, we see that multiplying by x to a product of $4n+3$ prime powers just multiplies the difference in $4n+1$ and $4n+3$ powers by the number of divisors of x. So we just need x to have 6 divisors (5^5 works) and the $4n+3$ powers to be even. So a possible answer is $\boxed{5^5 3^2 7^2}$.

8. Consider the graph $y = x^{3/2}$. $\lfloor i^{3/2} \rfloor$ is the number of lattice points below the graph at $x = i$ (includes the point on the graph if $i^{3/2}$ is an integer. But $y = x^{2/3}$ is the inverse of $y = x^{3/2}$. So $\lfloor i^{2/3} \rfloor$ is the number of lattice points to the left of the graph of $y = x^{3/2}$ at $y = i$. Hence overall since the bounds match up, this is just the area of a rectangle with side lengths 100 and 1000 plus the number of lattice points on the graph which is just 10. Hence our answer is $100(1000) + 10 = \boxed{100010}$.

9. We find the probability that $A \subseteq B$. The probability that an element is in A but not in B is $\frac{1}{4}$. So the probability that $A \subseteq B$ is $\frac{3^10}{4^10}$. But we are overcounting the probablility that $A = B$. The probability that an element is in A and in B is $\frac{1}{2}$. So probability that $A = B$ is $\frac{1}{2^10}$. So overall probability is $\boxed{\frac{2(3^{10}) - 2^{10}}{4^{10}}}$.

10. The answer is 21. Suppose there are more than 21 teams. Let the teams be $A_1, \ldots, A_k$. Then $|A_i| = 5$ and $|A_i \cap A_j| = 1$. Consider the intersection of A_1 with $A_2, \ldots, A_k$. Then some element of A_1 must appear at least 5 times by pigeonhole principle. Hence we have at least six teams sharing a person a. Call these teams $B_1, B_2, B_3, B_4, B_5, B_6$. Let b be a person on B_1 and b' in B_2. This team must intersect B_3, B_4, B_5, B_6 but this is not possible since the only element these sets share is a. So we can have at most 21 teams.

    On the other hand, if there are less than 21 teams, some person a can be on at most 4 teams. Suppose these are B_1, B_2, B_3, B_4. Take some person b from B_1 and b' from B_2. Then this team must share a teammate with B_3 and B_4. The last teammate c cannot come from B_1, B_2, B_3, B_4 otherwise two teams have the same pair of people. But then a and c are never on a team.

    So the answer is $\boxed{21}$.